$r(x)$ does not give the 'lateral extent' of the compressibility effects but is simply a measure of the initial rate of decay of the velocity $\bar{u}(\bar{x},\bar{z})$ with $\bar{z}$. Nørstrud's approximation to $r(\bar{x})$, Eq. (15), is not too inaccurate then it is not surprising that the comparison with Ref. 7 is qualitatively correct for $|\bar{z}/r(\bar{x})| < 1$. Indeed it would be surprising if it were not. In any event it is important in any integral equation method that the velocity in the neighborhood of the aerofoil is *quantitatively* correct in order to ensure adequate results in all cases, not just for the simple case of biconvex aerofoils.

The reason that Mach number plots were not included in Ref. 5 is that no comparable finite difference results, that is, flow in a freestream, were available in the literature; the case computed by Magnus et al.[7] being for a flow in a closed channel.

The third point that Nørstrud makes is his disagreement with the analysis of Ref. 5. No reason is given. It should be noted however that it is pointed out in Ref. 5 that Nørstrud's[3] basic integral equation *can* be written in the *form* of the first equation of Ref. 5. It is not suggested that the two equations are identical. The main point of the argument in Ref. 5 is that if the tangency boundary conditions are satisfied on the plane $z=\pm 0$ rather than on the aerofoil surface, then in the transonic integral equation there are two coupled nonlinear integral equations for the symmetric and anti-symmetric components of the flow, not one equations as implied in Ref. 3. A similar pair of equations, although uncoupled, are an established feature of thin aerofoil incompressible theory. Any transcription errors as regard signs in Ref. 5 do not obscure this fundamental point.

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Comment on "Gardon Heat Gage Temperature Response"

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KITA and Laganelli[1] have undertaken to show by numerical examples that the Gardon circular thin disk instrument for measuring heat transfer rate has properties more or less independent of varying edge temperature. The gage measurement of heat transfer rate in the steady state depends on the difference between the center temperature and the edge temperature. In the situation considered by Kita and Laganelli, the heat transfer rate was assumed constant but the edge temperature was varying. Further, in the numerical example presented in Ref. 1, the edge temperature variation with time was taken to be linear starting at time zero. For this edge temperature variation, the explicit analytical solution is available in the literature for an unheated cylindrical body and may be added by the principle of superposition to the solution for heat addition uniform over the disk imposed as a Heaviside step function beginning at time equals zero. The sum of these two solutions gives the complete analytical solution to the problem analyzed numerically in Ref. 1 and leads to somewhat different conclusions than those arrived at by the authors.

The solution[2] of the partial differential equation of heat conduction for a cylindrical body initially at constant temperature throughout (taken to be zero) and subjected to a temperature increasing as ϵt from $t=0$ gives for the temperature, T, in the body

$$T(r,t)=\epsilon\left[t-\frac{(a^2-r^2)}{4\kappa}\right]+\frac{2\epsilon}{a\kappa}\sum_{n=1}^{\infty}\frac{e^{-\kappa\alpha_n^2 t}J_0(r\alpha_n)}{\alpha_n^3 J_1(a\alpha_n)} \tag{1}$$

where r is the radial coordinate, a is the radius of the disk, κ is the thermal diffusivity of the material, $K/\rho c$, c is the specific heat, ρ is the density, and K is the thermal conductivity. J_n is the Bessel function of the first kind and of the nth order; the α_n's are the roots of the equation $J_0(a\alpha_n)=0$. Also, for the cylindrical body with edge temperature held at zero and subjected at $t=0$ to a step in heat transfer uniform over the interior, the solution to the heat conduction equation is[2]

$$T(r,t)=\frac{A_0(a^2-r^2)}{4K\delta}-\frac{2A_0}{aK\delta}\sum_{n=1}^{\infty}\frac{e^{-\kappa\alpha_n^2 t}J_0(r\alpha_n)}{\alpha_n^3 J_1(a\alpha_n)} \tag{2}$$

where A_0 is the rate of heat transfer to the surface of the disk, δ is the thickness of the disk, and other terms are as previously defined.

Adding these two solutions gives, by superposition, the solution to the Gardon gage problem with linear varying edge temperature as

$$T(r,t)-\epsilon t=\frac{(A_0-\rho c\delta\epsilon)(a^2-r^2)}{4K\delta}-\frac{2(A_0-\rho c\delta\epsilon)}{aK\delta}\sum_{n=1}^{\infty}e^{-\kappa\alpha_n^2 t}\frac{J_0(r\alpha_n)}{\alpha_n^3 J_1(a\alpha_n)} \tag{3}$$

Thus, through comparison of Eqs. (2) and (3), we arrive at the conclusion that the steady-state temperatures as well as the time variations of the temperature differences between the interior points of the disk and the edge with linear variation in edge temperature, ϵt, and with heat addition, A_0, are of the same functional form as for constant edge temperatures. However, the temperatures in the disk correspond to those which would result from an equivalent heat transfer rate to the surface equal to $A_0-\rho c\delta\epsilon$. Therefore, contrary to the conclusion reached by Kita and Laganelli, the measured heat transfer rate must, in general, be corrected for varying edge temperature. It is of course possible in specific cases that $\rho c\delta\epsilon$ may be sufficiently small relative to A_0 so that the correction may be neglected.

The close relationship between the temperature solutions for varying edge temperature and for heat input over the disk is not coincidental. Carslaw and Jaeger (Ref. 2, p. 294) show that in the typical heat conduction problem in which temperatures are arbitrarily specified at $t=0$, the temperatures may be considered to have been generated by instantaneous sources of heat of amount ρcT per unit volume, where T is taken relative to some reference temperature which is arbitrarily taken to be zero. In a more general case, the heating may be considered continuous of amount A' per unit volume such that $A'=\rho c dT/dt$. Moreover, if the reference value of

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temperature is continuously varied to be that of the edges, the continuous heating of the edge is precisely equivalent to continuous cooling of the interior of the body at the rate A'. This is the phenomenon exhibited by Eq. (3). By the same token, for other than a linear variation of T with time, A' is not represented by a step function at $t=0$, but varies continuously with time, and the corresponding temperature distribution in space and time is not given by Eqs. (1) or (2).

The general solution corresponding to Eq. (1) for a cylindrical body with edge temperature varying arbitrarily with time is also well known, being easily derived by Duhamel's principle from the solution for a step function in temperature at $t=0$ (edge temperature constant for $t>0$). If the varying edge temperature is given by $\varphi(t)$, then (from p. 176 of Ref. 2)

$$T(r,t)=\frac{2\kappa}{a}\sum_{n=1}^{\infty}e^{-\kappa\alpha_n^2 t}\frac{\alpha_n J_0(r\alpha_n)}{J_1(a\alpha_n)}\int_0^t e^{\kappa\alpha_n^2\tau}\varphi(\tau)\,d\tau \tag{4}$$

The integral may be integrated by parts to give

$$T(r,t)=\varphi(t)-\frac{2\varphi(0)}{a}\sum_{n=1}^{\infty}e^{-\kappa\alpha_n^2 t}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)}$$

$$-\frac{2}{a}\sum_{n=1}^{\infty}e^{-\kappa\alpha_n^2 t}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)}\int_0^t\frac{d\varphi(\tau)}{d\tau}e^{\kappa\alpha_n^2\tau}d\tau \tag{5}$$

where, to arrive at the first term on the right, use has been made of the identity (Ref. 2, p. 182).

$$1=\frac{2}{a}\sum_{n=1}^{\infty}\frac{J_0(r\alpha_n)}{\alpha_n J_1(a\alpha_n)} \tag{6}$$

Equations (4) or (5) allow Gardon gage results to be corrected for any time history of edge temperature, $\varphi(t)$, whatever.

Equation (1) for linear edge temperature variation is readily obtained from Eq. (5) by substituting $d\varphi(\tau)/d\tau=\epsilon, \varphi(0)=0$. The identification of the steady-state part of the solution requires some additional computation. Applying the Fourier-Bessel series expansion of r^2 (Ref. 2, p. 174)

$$r^2=\sum_{n=1}^{\infty}\frac{2}{a^2J_1^2(a\alpha_n)}J_0(\alpha_n r)\int_0^a\rho^3 J_0(\alpha_n\rho)\,d\rho \tag{7}$$

The integral is evaluated as

$$\int_0^a\rho^3 J_0(\rho\alpha_n)\,d\rho=\frac{a^3}{\alpha_n}J_1(a\alpha_n)-\frac{2a^2}{\alpha_n^2}J_2(a\alpha_n) \tag{8}$$

Using the recurrence formula for $J_n(a\alpha_n)$ we have[3]

$$2[J_1(a\alpha_n)/a\alpha_n]=J_0(a\alpha_n)+J_2(a\alpha_n) \tag{9}$$

We further note that $J_0(a\alpha_n)=0$ for all α_n in all cases considered here. Then, using Eq. (6) again,

$$-\frac{r^2-a^2}{4\kappa}=\sum_{n=1}^{\infty}\frac{2}{\kappa\alpha_n^3 a}\frac{J_0(r\alpha_n)}{J_1(a\alpha_n)} \tag{10}$$

which corresponds to the steady-state temperature in Eq. (1). For time variations other than linear of the edge temperature, $\varphi(t)$, it is obvious that other spatial variations of temperatures over the disk will result. Although the precedure is somewhat cumbersome, the steady-state part of the series solution can usually be summed in closed form by use of the Fourier-Bessel expansion theorem for elementary edge temperature functions as for the case of linear edge temperature variation discussed here.

In an earlier paper,[4] Ash and Wright presented, among other studies bearing on Gardon gage design and utilization, a Duhamel integral solution to the variable heat sink problem for the Gardon gage.† Their specific solution, however, was numerical and dealt with a heat sink temperature variation of the form $\varphi(t)=\epsilon t+\mu(1-e^{-\beta t})$ where t,μ, and β are constants, chosen in this case to fit experimental data. The first term, which is linear in time, has already been dealt with in Eq. (3); we need, therefore, only address the term multiplied by μ, which may be denoted by φ_1. Then $\varphi_1(0)=0$ and $d\varphi_1/dt=\mu\beta e^{-\beta t}$. Thus, the required solution to the equations of heat conduction corresponds to an exponential rate of heat transfer per unit of amount $T'=\rho c\mu\beta e^{-\beta t}$ per unit volume. The solution for this case is given by Ref. 2, p. 277.

$$T(r,t)-\varphi_1(t)=\frac{\kappa}{K\beta}(\rho c\mu\beta)e^{-\beta t}\left[\frac{J_0(r\sqrt{\beta/\kappa})}{J_0(a\sqrt{\beta/\kappa})}-1\right]$$

$$+2\frac{(\rho c\mu\beta)\kappa}{aK}\sum_{n=1}^{\infty}\frac{e^{-\kappa\alpha_n^2 t}J_0(r\alpha_n)}{\alpha_n(\kappa\alpha_n^2-\beta)J_1(a\alpha_n)} \tag{11}$$

where again the α_n are the positive roots of $J_0(a\alpha)=0$. By using the principle of superposition this solution may be added to Eq. (3) to give the combined effect of linear and exponential terms in the heat sink temperature variation.

If the exponential is slowly varying, the first term of Eq. (11), which is the quasi-steady-state term, may be expanded in powers of β/κ by using the power series for the Bessel function

$$J_0(x)=1-x^2/4+x^4/64+\ldots$$

to give

$$T(r,t)-\varphi_1=t)\approx-\rho c/K[\mu\beta e^{-\beta t}](a^2-r^2)/4 \tag{12}$$

Thus, with the rate of heat sink temperature change $\mu\beta e^{-\beta t}$ replacing ϵ, this is approximately the same expression as appears for the steady-state term in Eq. (1). A similar expansion in β/κ may be effected in the second term of Eq. (11), which is the transient part of the solution:

$$\frac{2\rho c\mu\beta}{aK}\sum_{n=1}^{\infty}\frac{e^{-\kappa\alpha_n^2 t}J_0(r\alpha_n)}{\alpha_n^3(1-\beta/\kappa\alpha_n^2)J_1(a\alpha_n)}=\frac{2\rho c}{aK}\mu\beta$$

$$\times\sum_{n=1}^{\infty}\frac{e^{-\kappa\alpha_n^2 t}J_0(r\alpha_n)}{\alpha_n^3 J_1(a\alpha_n)}\left(1+\frac{\beta}{\kappa\alpha_n^2}+\frac{\beta^2}{\kappa^2\alpha_n^4}+\ldots\right) \tag{13}$$

Here it is clear that it is necessary that the smallest value of α_n must be greater than $\sqrt{\beta/\kappa}$ for the series expansion to be valid. However, if $\alpha_1\gg\sqrt{\beta/\kappa}$ it can be seen that the transient part of the solution is substantially analogous to the transient part of Eq. (1) with the initial value of $d\varphi/dt=\mu\beta$ replacing the initial (and constant) value, ϵ, which corresponds in the case of linear temperature variation.

To obtain the quasi-steady-state term in Eq. (11) from Eqs. (4) or (5), it is only necessary to consider the Fourier-Bessel series expansion of $J_0(r\sqrt{\beta/\kappa})$, using the well-known formula

$$\int_0^a rJ_0(r\alpha_n)J_0(r\gamma)\,dr=\frac{a\alpha_n}{\alpha_n^2-\gamma^2}J_1(a\alpha_n)J_0(a\gamma)$$

where γ is an arbitrary constant and account has been taken of the fact that $J_0(a\alpha_n)=0$.

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†This paper was called to the writer's attention by one of the technical editors of the *AIAA Journal*.